irregularity is due to the term $3\text{W} \sin\theta \,.\, \text{S}_1 (2\omega t)/(2\omega t)^2$, which rapidly tends to zero.

Hence, for t large, the velocity of the liquid tangential to the sphere at a point of co-latitude θ, has components

$$-\tfrac{3}{2}\,\text{W} \sin\theta \cos(2\omega t \cos\theta), \qquad -\tfrac{3}{2}\,\text{W} \sin\theta \sin(2\omega t \cos\theta),$$

which combine into $\tfrac{3}{2}\text{W} \sin\theta$ in a direction making an angle α, measured in the negative direction from the meridian (θ increasing) through the point where $\alpha = 2\omega t \cos\theta$.

Hence, ultimately, the velocity of the liquid tangential to the sphere at any point is constant in magnitude and equal to $\tfrac{3}{2}\text{W} \sin\theta$, but changes its direction with constant angular speed $2\omega \cos\theta$, which is a maximum at the poles and zero at the equator.

A point to notice, however, is the fact that $(\partial v'/\partial\theta)_{r=a}$ contains the factor $2\omega t$, and therefore ultimately increases without limit. This means that we reach a stage beyond which the components of vorticity can no longer be regarded as small. A similar deduction is obtained from $\partial w/\partial r$ over the equatorial plane, which contains the factor $(2\omega t)^{\frac{1}{2}}$. A physical restriction to the solution of the problem is thus introduced.

On the Mutual Threading of Vortex Rings.

By W. M. HICKS, F.R.S.

(Received June 26, 1922.)

Although the fact that two circular vortices are capable of threading one another in permanent succession has long been known, I do not know that any attempt has been made at a discussion of the general conditions. The following pages contain a contribution to such a discussion for the case of thin circular filaments. The nature of a ring is given when its circulation and volume are known. Its configuration at any time is given when its aperture *or* the radius of the cross-section of the filament is known. In the case of two rings, the nature of their combination is defined if their two apertures at the instant when they are co-planar are each known. This configuration will be called their standard position. With two such rings, the mean area of the two apertures, supposed weighted with their circulations, remains constant throughout the motion. They can therefore also be defined by the radius of this mean area and the ratio of the two apertures in the standard position. This latter method is the more convenient. It is found possible to determine the relative paths of one as seen from the other, and to obtain the conditions of permanent union for any two given rings, but the complete solution in terms of the time, and the actual paths in the fluid, are not expressible in general terms. Such can only be arrived at by numerical quadrature for cases in which the various coefficients involved have their numerical values given. Any special case can thus be solved, but naturally the process is very laborious.

In the corresponding problem in two dimensions, however, a complete solution is attainable. For this reason, and because it illustrates the general method, the problem of two like and equal two-dimensional pairs (rectilinear filaments) is first touched upon. The more complete discussion for all the possible combinations is extremely interesting, especially when the pairs have opposite circulations, but is here omitted, in view of the greater physical importance of the theory of the ring.

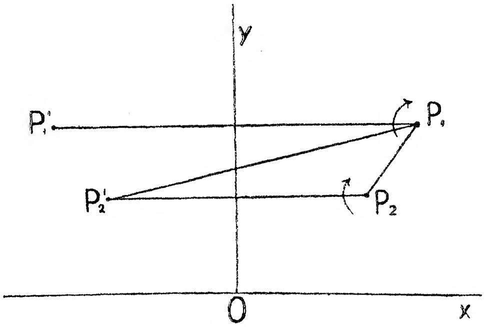


Fig. 1.

I.—*Parallel Straight Vortices.*

In the figure, P_1, P_1', P_2, P_2' denote the positions of the four filaments. Let x_1, y_1, x_2, y_2 denote the co-ordinates of P_1, P_2. Further, let x, y denote the co-ordinate of P_1 referred to P_2, *i.e.*,

$$x = x_1 - x_2, \qquad y = y_1 - y_2,$$

The stream functions of the pairs at a point ξ, η are given by $\mu_1\chi_1/4\pi$, $\mu_2\chi_2/4\pi$, where

$$\chi_1 = -\log\left(\frac{PP_1}{PP_1'}\right)^2 = -\log\frac{(\xi-x_1)^2+(\eta-y_1)^2}{(\xi+x_1)^2+(\eta-y_2)^2}. \tag{1}$$

It should be noted that the values of χ_1 at P_2 and of χ_2 at P_1 are the same.

Write $(PP_1/PP_1')^2 = 1-f_1^2$ where

$$f_1^2 = \frac{4\xi x_1}{(\xi+x_1)^2+(\eta-y_1)^2}$$

and let f denote the common value of f_1 at P_2 or of f_2 at P_1. Then, putting $\mu_1 = \mu_2 = \mu$

$$\left.\begin{aligned} \frac{dx_1}{dt} &= -\frac{\mu}{4\pi}\left.\overline{\frac{d\chi_2}{dy}}\right|_1 = \frac{\mu}{\pi}\frac{f^3}{4x_1x_2}y\frac{d\chi}{df} \\ \frac{dx_2}{dt} &= \dots \qquad\quad = -\frac{\mu}{\pi}\frac{f^3}{4x_1x_2}y\frac{d\chi}{df} \end{aligned}\right\}, \tag{2}$$

whence

$$x_1 + x_2 = \text{const.} = \mathrm{A},$$

$$\frac{dx}{dt} = \frac{\mu}{4\pi x_1 x_2}\frac{f^4}{1-f^2}y, \tag{3}$$

and

$$f^2 = \frac{\mathrm{A}^2 - x^2}{\mathrm{A}^2 + y^2}.$$

Also

$$\left.\begin{aligned} \frac{dy_1}{dt} &= \frac{\mu}{4\pi x_1} + \frac{\mu}{4\pi}\left.\overline{\frac{d\chi_2}{dx}}\right|_1 = \frac{\mu}{4\pi x_1} + \frac{\mu}{4\pi}\left(\frac{1}{2x_1} - \frac{x_1+x_2}{4x_1x_2}f^2\right)f\frac{d\chi}{df} \\ \frac{dy_2}{dt} &= \dots \qquad\qquad\qquad\quad = \frac{\mu}{4\pi x_2} + \frac{\mu}{4\pi}\left(\frac{1}{2x_2} - \frac{x_1+x_2}{4x_1x_2}f^2\right)f\frac{d\chi}{df} \end{aligned}\right\}. \tag{4}$$

Hence

$$\frac{dy}{dt} = -\frac{\mu}{4\pi x_1 x_2}\frac{1}{1-f^2}x, \tag{5}$$

$$\therefore \qquad \frac{dx}{dy} = -f^4\frac{y}{x} = -\frac{y}{x}\left(\frac{\mathrm{A}^2 - x^2}{\mathrm{A}^2 + y^2}\right)^2,$$

The integral of this is

$$\frac{1}{\mathrm{A}^2 + y^2} - \frac{1}{\mathrm{A}^2 - x^2} = \mathrm{B}. \tag{6}$$

2. We shall define a given configuration by the apertures when the two pairs are co-planar, say $2a$, $2b$, $a > b$. Then

$$\mathrm{A} = a + b, \qquad \mathrm{B} = \frac{1}{(a+b)^2} - \frac{1}{4ab}.$$

It follows from (6) that

$$y^2 = \frac{(a+b)^4}{(a-b)^2}\frac{(a-b)^2 - x^2}{x^2 + (a+b)^2\{4ab/(a-b)^2 - 1\}}.$$

If, then, $4ab > (a-b)^2$, y is always finite, or the filaments revolve round each other in permanent union. If $4ab < (a-b)^2$, one pair simply passes through the aperture of the other, and then separates to infinity. The above condition gives for permanent union

$$a > (3 + 2\sqrt{2})\,b, \qquad > 5{\cdot}8284b,$$
$$b < 0{\cdot}17157a.$$

Equation (6) may also be written

$$x^2 = (a+b)^2\frac{(a^2-b^2)^2 + \{(a-b)^2 - 4ab\}\,y^2}{(a+b)^4 + (a-b)^2 y^2}.$$

In this put

$$\frac{(a^2-b^2)^2}{4ab-(a-b)^2} = \alpha^2, \qquad \frac{(a+b)^4}{(a-b)^2} = \beta^2.$$

Then

$$x^2 = \frac{(a+b)^4}{\alpha^2}\,\frac{\alpha^2-y^2}{\beta^2+y^2}. \tag{7}$$

3. *Period of Revolution.*—From (5) since $4x_1x_2 = A^2-x^2$,

$$\frac{dy}{dt} = -\frac{\mu}{\pi}\,\frac{A^2+y^2}{x^2+y^2}\cdot\frac{x}{A^2-x^2}.$$

By (7) it can be shown that

$$\frac{dy}{dt} = -\frac{\mu}{\pi}\,\frac{(a-b)^2}{4ab\alpha}\left(\frac{\beta^2+y^2}{A^2+y^2}\right)^2\sqrt{\frac{\alpha^2-y^2}{\beta^2+y^2}}.$$

Put $y = \alpha\cos\phi$ and $k^2 = \alpha^2/(\alpha^2+\beta^2) = (a-b)^4/16a^2b^2$ so that

$$k = \frac{(a-b)^2}{4ab}.$$

Then

$$\frac{dt}{d\phi} = \frac{\pi(a+b)^2}{\mu k(1-k)}\left\{\sqrt{(1-k^2\sin^2\phi)}-\frac{2(1-k)}{\sqrt{(1-k^2\sin^2\phi)}}+\frac{(1-k)^2}{(1-k^2\sin^2\phi)^{3/2}}\right\}$$

whence integrating

$$t = \frac{2\pi(a+b)^2}{\mu k}\left\{\frac{1}{k'^2}\,\mathrm{E}(\phi)-\mathrm{F}(\phi)-\tfrac{1}{2}\,\frac{k^2}{1+k}\,\frac{\sin\phi\cos\phi}{\sqrt{(1-k^2\sin^2\phi)}}\right\}. \tag{8}$$

The period of a complete revolution is therefore

$$\mathrm{T} = \frac{8\pi(a+b)^2}{\mu k}\left(\frac{1}{k'^2}\,\mathrm{E}-\mathrm{F}\right) = \frac{8\pi(a+b)^2}{\mu}\,\frac{d\mathrm{F}}{dk}. \tag{8a}$$

Thus, for a given ratio of apertures, the period varies as the square of their linear values.

Expressed in terms of the angle ϕ, the relative co-ordinates are

$$\left.\begin{aligned} x &= (a+b)\surd\,.\,k(1-k)\,\frac{\sin\phi}{\sqrt{(1-k^2\sin^2\phi)}} \\ y &= (a+b)\sqrt{\left(\frac{k}{1-k}\right)}\cos\phi \end{aligned}\right\}. \tag{9}$$

4. *Motion in the Fluid.*—The relative motion is completely determined by the preceding results. To determine the actual motion in the fluid itself, it is only necessary to determine further the motion of the centre of gravity in the direction of translation.

If Y denote its position at time t,

$$2\mathrm{Y} = y_1 + y_2,$$

$$2\frac{d\mathrm{Y}}{dt} = \frac{\mu(x_1+x_2)}{4\pi x_1 x_2}(1+f^2) \text{ from (4, 5)}$$

and

$$2\frac{d\mathrm{Y}}{dy} = -\mathrm{A}(1-f^4)\frac{1}{x}.$$

Substituting for x, y their values in terms of ϕ from (9)

$$2\frac{d\mathrm{Y}}{d\phi} = \frac{a+b}{1-k}\left\{-\sqrt{(1-k^2\sin^2\phi)} + \frac{(1-k)^2}{(1-k^2\sin^2\phi)^{3/2}}\right\},$$

$$2\mathrm{Y} = (a+b)\frac{k}{k'^2}\left\{2\mathrm{E}(\phi) - k(1-k)\frac{\sin\phi\cos\phi}{\sqrt{(1-k^2\sin^2\phi)}}\right\}.$$

Hence the total distance traversed during a complete revolution is

$$4(a+b)\frac{k}{k'^2}\mathrm{E}$$

and is independent of the strength of the vortices.

The path of a filament is then given by

$$x' = x_1 = \tfrac{1}{2}(a+b+x),$$

$$y' = \mathrm{Y} + \tfrac{1}{2}y.$$

The mean velocity of propagation of the system through the fluid is

$$\frac{\mathrm{Y}}{\mathrm{T}} = \frac{\mu}{2\pi(a+b)}\frac{k^2\mathrm{E}}{\mathrm{E}-k'^2\mathrm{F}}.$$

Since $k'^2\mathrm{F} > (1-k^2)\mathrm{E}$, this mean velocity is greater than $\mu/[2\pi(a+b)]$, the velocity of progression of a single pair by themselves of aperture $a+b$. Also, as $k^2\mathrm{E}/(\mathrm{E}-k'^2\mathrm{F})$ is less than 2, this mean velocity is less than twice the latter, as, indeed, is evident from general considerations. In fact, the mathematically limiting case corresponds to that in which, while $a-b$ is finite, $a+b$ is infinite, when the two velocities become equal.

5. *Graphical Construction.*—The co-ordinates of the relative orbit may be written

$$y = c\cos\phi,$$

$$x = \frac{1-k}{k}c\frac{c\sin\phi}{\sqrt{(c^2/k^2 - c^2\sin^2\phi)}},$$

where $c = (a+b)\sqrt{(k/\,.\,1-k)}$. They suggest the following graphical method for tracing the curves.

Take two rectangular axes Ox, Oy. Draw three concentric circles (centre O) of radii, c, $c(1-k)/k$, and c/k. The last is clearly the largest and the second is larger than the first so long as $k < \frac{1}{2}$. Take any point P on the first and draw PM perpendicular to Ox cutting the outer circle in R. Join OR cutting the second circle in Q, and draw QT perpendicular to OQ, meeting Oy in T. Then

$$\text{angle POT} = \phi$$
$$\text{PM} = y$$
$$\text{QT} = x$$

The construction is simplified if $k = \frac{1}{2}$ since then the first and second circles coincide, and the third has twice their radius.

6. The foregoing results are illustrated by application to two special cases of $k = \sin 10°$ and $\sin 60°$. These correspond to ratios of aperture in the standard co-planar positions of $b/a = 0{\cdot}444$ and $0{\cdot}189$. The relative paths are given in fig. 2 when the sum $a+b$ is the same. Curve A gives that for $b/a = 0{\cdot}189$ (that nearer the limiting case) and curve B that for $b/a = 0{\cdot}444$. The dots on the curves give the positions at equal intervals of time, equal to one-sixteenth of the total period. They serve to show how the velocity varies along the path. It is especially noticeable how in the case of small b/a—curve A—the motion becomes very slow when the apertures approach equality. The corresponding paths in the fluid are given by figs. 3 and 4 for the right hand components, the thick line curve belonging to one component and the thin to the other. The paths described by the left hand components correspond (right and left reversed) at distances on the left of those in the figures respectively equal to 1·47 and 2·60 times the distance 4 ... 4 in the co-planar position. The dots on these curves again give positions at equal intervals. The curves are scrolls, each curve intersecting itself. It might be thought that possibly, for some critical state, the small loop might degenerate to a cusp, followed by states with sinuous paths. The condition would be given by making $dy_1/dt = 0$ when $y = 0$. It would be found that this corresponds to the critical state for permanent union. In other words the paths are always scrolls when in permanent union and sinuous when not.

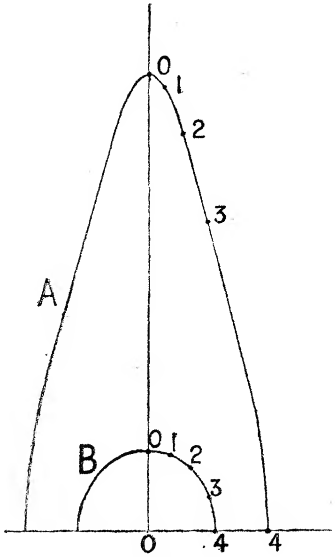


FIG. 2.

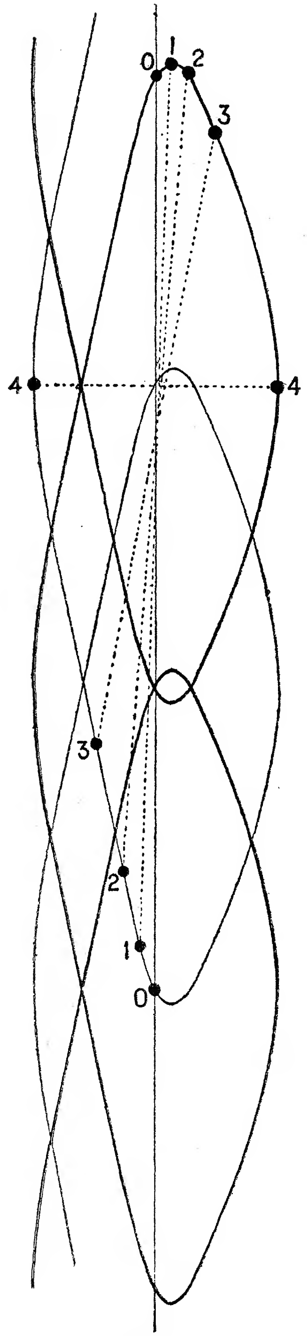


Fig. 3.

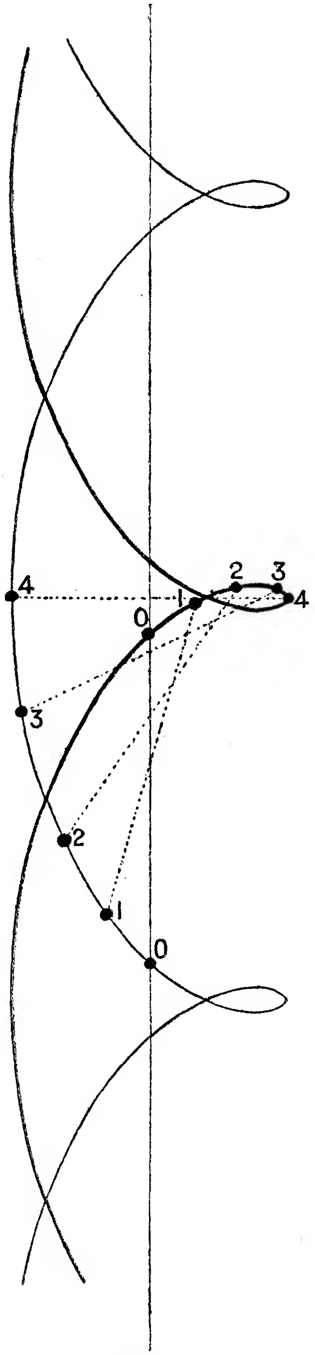


Fig. 4.

II. *Two Circular Rings.*

7. We now pass to our proper problem of the motion of two circular vortex filaments with the same straight axis. In the general case the circulations will be different, say μ_1, μ_2. Let x_1, x_2 denote the radii of their circular axes at any time and y_1, y_2 the distances of their planes from a plane of reference. The stream function for a single circular vortex (x_1, y_1) is given by

$$\frac{\mu}{\pi}\chi_1 = \sqrt{(xx_1)}f_1,$$

where

$$f_1 = \frac{1+k'^2}{2k}\,\mathrm{F} - \frac{1}{k}\,\mathrm{E}, \tag{1}$$

in which

$$k^2 = \frac{4x_1x}{(x+x_1)^2+(y-y_1)^2}. \tag{2}$$

Also the velocity of progression due to its own constitution is

$$\mathrm{U} = \frac{\mu}{4\pi x_1}\left(\log_e\frac{8x_1}{a} - \frac{1}{4}\right)$$

where a is the radius of the cross-section of the filament. With varying aperture this will alter so as to keep the volume of the filament constant. If this be denoted by $m = 2\pi x\,.\,\pi a^2$

$$\mathrm{U} = \frac{\mu}{8\pi x}\left(\log_e\frac{128\pi^2}{m}x^3 - \frac{1}{2}\right)$$

$$= \frac{3\mu}{8\pi x}\log_e \lambda x, \tag{3}$$

where

$$\lambda = \left(\frac{128\pi^2}{m\sqrt{e}}\right)^{1/3}.$$

In what follows we shall omit the π and suppose it contained in the μ. It will be noticed that, as in the case of the straight filaments, $\overline{\chi_1}|_2 = \overline{\chi_2}|_1$ and k is the same quantity as that denoted by f in the treatment of the two-dimensional case. Hence

$$\frac{dk}{dx_1} = \frac{k}{2x_1} - \frac{k^3(x_1+x_2)}{4x_1x_2}, \qquad \frac{dk}{dy_1} = -\frac{k^3(y_1-y_2)}{4x_1x_2},$$

$$\frac{dk}{dx_2} = \frac{k}{2x_2} - \frac{k^3(x_1+x_2)}{4x_1x_2}, \qquad \frac{dk}{dy_2} = \frac{k^3(y_1-y_2)}{4x_1x_2}.$$

It follows as before, writing y for y_1-y_2 that

$$\left.\begin{aligned} \frac{dx_1}{dt} &= \frac{\mu_2}{x_1}\left.\overline{\frac{d\chi_2}{dy}}\right|_1 = \mu_2\frac{k^3}{4x_1^2x_2}\,y\,\frac{d\chi}{dk} \\ \frac{dx_2}{dt} &= \ldots \qquad\quad = -\mu_1\frac{k^3}{4x_1x_2^2}\,y\,\frac{d\chi}{dk} \end{aligned}\right\}, \tag{4}$$

whence
$$\mu_1 x_1 \frac{dx_1}{dt} + \mu_2 x_2 \frac{dx_2}{dt} = 0,$$

or
$$\mu_1 x_1^2 + \mu_2 x_2^2 = \mathrm{A} = \mu_1 a^2 + \mu_2 b^2, \tag{5}$$

where a, b are the radii of the apertures in the co-planar position.

We can therefore express x_1, x_2 in terms of one variable θ, where

$$x_1 = \sqrt{\frac{\mathrm{A}}{\mu_1}} \cos\theta, \qquad x_2 = \sqrt{\frac{\mathrm{A}}{\mu_2}} \sin\theta.$$

Further, write $\mu_1 = \mu\cos^2\alpha$, $\mu_2 = \mu\sin^2\alpha$ and let d denote the radius of the mean aperture of the rings, weighted by their respective circulations, that is

$$(\mu_1 + \mu_2)\, d^2 = \mu_1 a^2 + \mu_2 b^2.$$

Then
$$\mathrm{A} = (\mu_1 + \mu_2)\, d^2 = \mu d^2,$$
$$x_1 = d\cos\theta/\cos\alpha, \qquad x_2 = d\sin\theta/\sin\alpha,$$
$$x_1 x_2 = d^2 \sin 2\theta/\sin 2\alpha, \qquad x = x_1 - x_2 = 2d\sin(\alpha - \theta)/\sin 2\alpha.$$

Again write
$$\mu_1 x_1^2 - \mu_2 x_2^2 = \mathrm{X},$$

so that
$$\mathrm{X} = \mu d^2 \cos 2\theta \text{ and}$$
$$\frac{d\mathrm{X}}{dt} = \mu_1\mu_2 \frac{k^3 y}{x_1 x_2} \frac{d\chi}{dk}. \tag{6}$$

Again
$$\frac{dy_1}{dt} = \frac{3\mu_1}{8x_1} \log \lambda_1 x_1 + \frac{\mu_2}{x_1} \left.\overline{\frac{d\chi_2}{dx}}\right|_1, \qquad \frac{dy_2}{dt} = \frac{3\mu_2}{8x_2} \log \lambda_2 x_2 + \frac{\mu_1}{x_2} \left.\overline{\frac{d\chi_1}{dx}}\right|_2.$$

Write
$$\phi(\mathrm{X}) = \frac{3\mu_1}{8x_1} \log \lambda_1 x_1 - \frac{3\mu_2}{8x_2} \log \lambda_2 x_2$$
$$= \frac{3\mu}{8d} \left\{ \frac{\cos^2\alpha}{\cos\theta} \log\left(\lambda_1 d \frac{\cos\theta}{\cos\alpha}\right) - \frac{\sin^2\alpha}{\sin\theta} \log\left(\lambda_2 d \frac{\sin\theta}{\sin\alpha}\right) \right\}.$$

Then
$$\frac{dy}{dt} = \phi(\mathrm{X}) + \frac{\mu_2}{x_1}\frac{d\chi}{dx_1} - \frac{\mu_1}{x_2}\frac{d\chi}{dx_2}$$
$$= \phi(\mathrm{X}) - \tfrac{1}{2}\frac{\mathrm{X}}{x_1^2 x_2^2}\chi + \left\{ -\tfrac{1}{2}\frac{\mathrm{X}}{x_1^2 x_2^2} + \frac{x_1 + x_2}{4x_1^2 x_2^2}(\mu_1 x_1 - \mu_2 x_2) k^2 \right\} k \frac{d\chi}{dk}.$$

Further
$$\frac{dk}{dt} = \frac{dk}{dx_1}\frac{dx_1}{dt} + \dots$$
$$= \tfrac{1}{2} k \left(\frac{1}{x_1}\frac{dx_1}{dt} + \frac{1}{x_2}\frac{dx_2}{dt} \right) - \frac{k^3 (x_1 + x_2)}{4x_1 x_2} \left(\frac{dx_1}{dt} + \frac{dx_2}{dt} \right) - \frac{k^3 y}{4x_1 x_2}\frac{dy}{dt}.$$

Substituting the values for dx/dt, dy/dt given above, it can be shown that

$$\frac{dk}{dt} = \left\{ \frac{\mathrm{X}}{8x_1^3 x_2^3}\chi - \frac{1}{4x_1 x_2}\phi(\mathrm{X}) \right\} k^3 y.$$

Substituting for k^3y/x_1x_2 from (6)

$$\frac{d\chi}{dk}\frac{dk}{d\mathrm{X}} = \frac{1}{4\mu_1\mu_2}\left\{\frac{\mathrm{X}}{2x_1^2x_2^2}\chi - \phi(\mathrm{X})\right\},$$

$$\sqrt{(x_1x_2)}\frac{df}{d\mathrm{X}} = \frac{1}{4\mu_1\mu_2}\left\{\frac{\mathrm{X}}{2(x_1x_2)^{3/2}}f - \phi(\mathrm{X})\right\},$$

or transforming to θ

$$\sqrt{\left(\frac{\sin 2\theta}{\sin 2\alpha}\right)}\frac{df}{d\theta} + \frac{\cos 2\theta}{\sqrt{(\sin 2\alpha \sin 2\theta)}}f = \frac{2d}{\mu}\frac{\sin 2\theta}{\sin^2 2\alpha}\phi(\theta),$$

$$\frac{d}{d\theta}(f\sqrt{\sin 2\theta}) = \frac{3}{8\sqrt{2}}\left\{\cot^{3/2}\alpha \sin\theta \log \lambda_1 d\frac{\cos\theta}{\cos\alpha} - \tan^{3/2}\alpha\cos\theta\log\lambda_2 d\frac{\sin\theta}{\sin\alpha}\right\}.$$

Hence

$$\begin{aligned} f\sqrt{\sin 2\theta} &= \mathrm{C} - \frac{3}{8\sqrt{2}}\{\cot^{3/2}\alpha\cos\theta\log_e(\mathrm{P}_1\cos\theta) \\ &\qquad + \tan^{3/2}\alpha\sin\theta\log_e(\mathrm{P}_2\sin\theta)\} \\ &= \mathrm{C} - 0{\cdot}61108\{\cot^{3/2}\alpha\cos\theta\log_{10}(\mathrm{P}_1\cos\theta) + \ldots\}, \qquad (7) \end{aligned}$$

where $\mathrm{P}_1 = \lambda_1 d/(e\cos\alpha), \qquad \mathrm{P}_2 = \lambda_2 d/(e\sin\alpha).$

If c denotes the radius of a sphere whose volume is that of a ring

$$\lambda = \left(\frac{128\pi^2}{m\sqrt{e}}\right)^{1/3} = \frac{1}{c}\left(\frac{96\pi}{\sqrt{e}}\right)^{1/3}$$

and $$\mathrm{P}_1 = \frac{2{\cdot}9533}{\sqrt{2}\cos\alpha}\frac{d}{c_1}, \qquad \mathrm{P}_2 = \frac{2{\cdot}9533}{\sqrt{2}\sin\alpha}\frac{d}{c_2}. \qquad (7a)$$

When the rings have equal circulations,

$$\mathrm{P}_1 = \mathrm{P}_2 = 2{\cdot}9533\, d/c.$$

In all cases where our present supposition of thin filaments holds, d is always a very large multiple of c, of the order 100 or larger. It will be convenient to suppose it given by 10^n. It corresponds to a cross-section whose radius $= 2{\cdot}122 \times 10^{-(3n+1)/2}d$.

The equation to the relative path may be written in any given case

$$\begin{aligned} f\sqrt{\sin 2\theta} = \mathrm{C} - \mathrm{L}\cos(\theta-\gamma) &- 0{\cdot}61108\cot^{3/2}\alpha\cos\theta\log\cos\theta \\ &- 0{\cdot}61108\tan^{3/2}\alpha\sin\theta\log\sin\theta. \qquad (8) \end{aligned}$$

The values of the functions $0{\cdot}61108\cos\theta\log_{10}\cos\theta$, $0{\cdot}61108\sin\theta\log_{10}\sin\theta$ are given at intervals of 5° from 0 to 45 in Table I. It is not necessary to give them for higher values since for $\pi/2-\theta$ they interchange their values. The values of f are given in Table II. In calculating any path, then, f should first be represented by a curve drawn on squared paper with abscissæ $= \phi$, where $k = \sin\phi$. This should be done once for all, and serves for the determination of any path. If the right-hand side of the equation is divided by $\sqrt{\sin 2\theta}$, the path is determined by an equation of the form

$$f(\phi) = \mathbf{f}(\theta).$$

Then $\mathbf{f}(\theta)$ is calculated for any θ. The f curve then gives the value of ϕ for which $f(\phi)$ has this value. Then x_1, x_2, $x = x_1 - x_2$ are given by θ, and y is found from $k = \sin\phi$ by

$$k^2 = \frac{4x_1x_2}{(x_1+x_2)^2+y^2},$$

or

$$y = \frac{2d}{\sin 2\alpha}\left\{\frac{\sin 2\theta \sin 2\alpha}{k^2} - \sin^2(\theta+\alpha)\right\}^{1/2}$$

and

$$x = \frac{2d}{\sin 2\alpha}\sin(\alpha-\theta).$$

The foregoing discussion is completely general. In the application, however, to actual cases a modification is required when the two circulations are oppositely directed. We take first the case when these are the same.

8. *Conditions for Permanent Union* (μ_1, μ_2 same sign).—We may suppose any given arrangement of the rings defined by their apertures at the instant when the rings are co-planar. If θ_0 be the value of θ at this instant

$$b/a = \tan\theta_0 \cot\alpha.$$

At this instant, $y = 0$ and

$$\cos\phi = k' = \frac{a-b}{a+b},$$

$$\therefore \quad \frac{b}{a} = \frac{1-\cos\phi}{1+\cos\phi} = \tan^2\tfrac{1}{2}\phi = \tan\theta_0\cot\alpha, \tag{9}$$

so that the value of f (say f_0) at this instant is determinable in terms of θ_0. Putting in these values in equation (7) determines the value of the constant C for the given arrangement.

There will be some critical value of θ_0 below which one ring passes through the other and separates to infinity, and above which the two will remain in permanent union, continually threading each other. To determine this, consider the arrangement in the critical case when, say, ring II was at an infinite distance before reaching I. The two rings must now have equal velocities of progression. If that of II were just less it would never reach I. If it were just greater it would ultimately reach I, and since the paths are symmetrical, pass off to infinity at the other side. Thus the critical state (θ_0) corresponds to one in which the two rings have equal velocities of progression when they have separated so far as to be beyond each other's direct influence.

If their radii are then x_1, x_2, this condition gives

$$\frac{\mu_1}{x_1}\log\lambda_1 x_1 = \frac{\mu_2}{x_2}\log\lambda_2 x_2,$$

or

$$\tan\theta\,\frac{\log(\mathrm{P}_1 e\cos\theta)}{\log(\mathrm{P}_2 e\sin\theta)} = \tan^3\alpha. \tag{10}$$

Let θ_1 be the solution of this equation. It must also satisfy the equation to the path. Now, at infinity $k = 0$ and $f = 0$. Hence

$$\mathrm{C} = \mathrm{L}\cos(\theta_1-\gamma) + 0{\cdot}61108\cot^{3/2}\alpha\cos\theta_1\log\cos\theta_1 + \ldots .$$

The critical value θ_0 is therefore the root of the equation

$$f_0\sqrt{\sin 2\theta_0} = \mathrm{L}\cos(\theta_1-\gamma) + 0{\cdot}61108\cot^{3/2}\alpha\cos\theta_1\log\cos\theta_1 + \ldots$$
$$-\mathrm{L}\cos(\theta_0-\gamma) - 0{\cdot}61108\cot^{3/2}\alpha\cos\theta_0\log\cos\theta_0 - \ldots . \tag{11}$$

When the circulations are equal, the solution of (10) is $\theta_1 = \frac{1}{4}\pi$, as it clearly should be since then $b = a$.

9. *Case of Equal Rings.*—Here $\mu_1 = \mu_2$, $\alpha = \pi/4$, $\gamma = \pi/4$, and $c_1 = c_2$. Also $\theta_1 = \pi/4$. The critical equation is now

$$f_0\sqrt{\sin 2\theta_0} = 0{\cdot}86419n + 0{\cdot}27636 - (0{\cdot}86419n + 0{\cdot}40603)\cos(\pi/4-\theta_0)$$
$$-0{\cdot}61108(\cos\theta_0\log\cos\theta_0 + \sin\theta_0\log\sin\theta_0), \tag{11a}$$

f_0 is calculated as a function of θ_0 since $\tan^2\frac{1}{2}\phi = \tan\theta_0$. The curves are drawn for η = left-hand side and η = right-hand side. It is only necessary to do this for $\theta < 45°$, since now the curves are the same for θ and $\pi/2-\theta$. The first curve increases continuously from 0 to ∞, the second decreases from a finite positive value to zero. Hence they must intersect at a single point which uniquely determines θ_0. For the case of $n = 3$, or $d = 1000\,c$, this root is given by $\theta = 23° \, 30'$. Hence the critical ratio of the apertures is given by

$$b/a = \tan 23°\,30' = 0{\cdot}4348.$$

With equal rings in which $d = 1000\,c$ then the ratio of the apertures, when co-planar, must be greater than 0·4348, if they are to remain in permanent union.

The above analysis may be illustrated by considering two arrangements, one near the critical state, the other considerably greater, when $d = 1000\,c$. The relative paths for $b/a = \frac{1}{2}$ and $\frac{3}{4}$ or $\theta_0 = 26°\,34'$ and $36°\,52'$ are represented by curves A and B in fig. 5.

The foregoing treatment of equation (11a), to find the critical ratio, when d/c is given, is, however, too restricted. The equation shows that any value of the critical ratio is possible provided the absolute values of the apertures, or of d, are properly chosen. In fact, the equation may be written

$$n+0{\cdot}4698 = 1{\cdot}1571\frac{f\sqrt{\sin 2\theta_0}+0{\cdot}1300+0{\cdot}61108\,(\cos\theta_0\log\cos\theta_0+\dots)}{1-\cos(\pi/4-\theta_0)}.$$

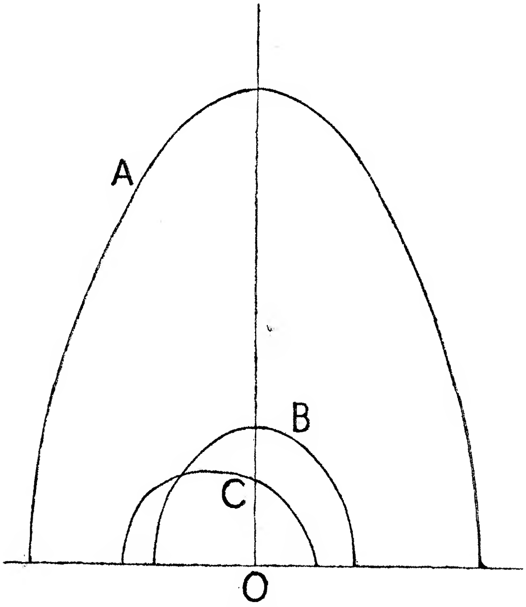


Fig. 5.

The numerator of the fraction on the right is always positive, since the maximum of the negative terms in logs is 0·130 when $\theta_0=\pi/4$. As θ_0 increases from 0 to $\pi/4$, the denominator decreases from $1-1/\sqrt{2}$ to 0, whilst the $f\sqrt{\sin 2\theta_0}$ increases from 0 to ∞. Hence $n+0{\cdot}4698$ increases from $1{\cdot}1571\times 0{\cdot}1300\,(2+\sqrt{2})=0{\cdot}5138$ to ∞, or n from 0·44 to ∞. But in order that our stream function may be valid, not only must d/c be large, but the smaller aperture, $b=d\sqrt{2}\sin\theta_0$, must be large compared with the radius of its cross-section. As an extreme case for the formula validity, take this ratio >10, which corresponds to $b>10^{0{\cdot}551}c$, whence

$$d\sqrt{2}\sin\theta_0>10^{0{\cdot}551}c,$$

$$n>0{\cdot}551+\log\operatorname{cosec}\theta_0-0{\cdot}150>0{\cdot}400+\log\operatorname{cosec}\theta_0.$$

From the equation we find

θ_0.	n.
10°	0·0548
15°	0·4456
20°	1·432
25°	4·211

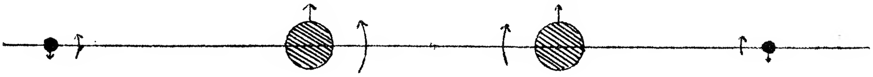
Hence n is too small for 10°, 15°. For $\theta = 20°, 0·400 + \log \operatorname{cosec} 20° = 0·867$, so that for this angle, and above, the value of n comes within the limits of formula validity. The limit is found close to 18°, for which $n = ·910$ and $0·400 + \log \operatorname{cosec} 18° = 0·917$. This gives

$$b/a = 0·325\,; \quad d/c = 10^{0·910}\,; \quad b/a = 10^{0·360}\,; \quad a/c = 10^{1·038},$$

whence if r denote the radius of cross-section of the filament

$$a/r_1 = 52·6, \qquad b/r_2 = 5·17.$$

This system is drawn to scale in the following diagram—



The configuration is clearly one in which the assumption of a thin filament does not hold, and the stream function is no longer approximate. But it is sufficiently close to show that the least possible ratio of b/a for permanent union is close to tan 19°, or (say) one-third, with a mean aperture given by $d = 10\,c$. If a less ratio is possible, it must be accompanied by considerable deformations of the cross-section of the inner, which also must have considerable thickness.

If the ratio b/a for the limiting case approaches unity, n is exceedingly large. Are the calculations now invalid, by a too close approximation of the two rings? To answer this, take $\theta_0 = \pi/4 - \theta$, with θ tending to zero. Then

$$a = d\sqrt{2}\cos(\pi/4 - \theta) = d(1 + \theta),$$
$$b = d\sqrt{2}\sin(\pi/4 - \theta) = d(1 - \theta),$$
$$a - b = 2d\theta,$$

also the cross-sectional radii will tend to equality, say r, where

$$r = 2·122 \times 10^{-(3n+1)/2}\,d,$$

$$\therefore \quad (a - b)/r = \theta\, 10^{(3n+1)/2},$$

also n is now of the order

$$n = \frac{f}{\frac{1}{2}\theta^2} = \frac{2f}{\theta^2} = \frac{1}{\theta^2}\log\sec\phi$$

and
$$\tan\frac{\phi}{2} = \sqrt{\tan\left(\frac{\pi}{4} - \theta\right)} = 1 - \theta.$$

Hence
$$\phi = \frac{\pi}{2} - \theta \quad \text{and} \quad n = \frac{1}{\theta^2}\log\frac{1}{\theta}.$$

Thus
$$\frac{a-b}{r} \text{ is of order } \theta\left(\frac{1}{\theta}\right)^{3/\theta^2},$$

or the distance of the rings from one another is infinitely larger than the cross-sections of the filaments. Consequently the above results are valid when a, b approach equality.

10. *Case of Different Circulations.*—In choosing illustrations for this kind of motion, the volumes of the rings, as well as their circulations, have to be taken account of. We shall for this purpose confine our attention to configurations where the vortical matter is of the same quality, or the vorticity is the same. In this case, if ω denote the vortical rotation at any point in a ring at a distance x from the straight axis, $\omega = kx$, where k is a constant. Hence

$$\mu = 2\pi\omega r^2 = 2\pi kxa^2 = \frac{k}{\pi}m = \frac{4}{3}kc^3.$$

Thus $$\cot^2\alpha = \mu_1/\mu_2 = m_1/m_2 = (c_1/c_2)^3.$$

Take c to represent the radius of that sphere whose diametral area is the sum of those of the rings, so that

$$c^2 = c_1{}^2+c_2{}^2$$

and put $$c_1 = c\cos\beta, \qquad c_2 = c\sin\beta.$$

Then $\cot^2\alpha = \cot^3\beta$, and equations (7$a$) become

$$\mathrm{P}_1 = \frac{2\cdot9533}{\sqrt{2\cos\alpha\cos\beta}}\frac{d}{c}, \qquad \mathrm{P}_2 = \frac{2\cdot9533}{\sqrt{2\sin\alpha\sin\beta}}\frac{d}{c}.$$

For a case in illustration take $d = 1000\,\epsilon$ and $\cot^{3/2}\alpha = 3$. This gives

$$\alpha = 25^\circ\ 40'\ 34'', \qquad \beta = 31^\circ\ 32'\ 11'',$$

$$\mu_1/\mu_2 = \cot^2\alpha = 4\cdot3267,$$

$$c_1/c_2 = 1\cdot6295,$$

$$\log\mathrm{P}_1 = 3\cdot4343, \qquad \log\mathrm{P}_2 = 3\cdot9645.$$

In order to determine the condition of permanent union, equation (10) is found to be

$$\log\left\{\tan\theta\,\frac{3\cdot8686-\log\sec\theta}{4\cdot3988-\log\operatorname{cosec}\theta}\right\} = 1\cdot04576.$$

The root is $\theta_1 = 5^\circ\ 33'\ 18''$. This makes the critical constant $\mathrm{C} = 6\cdot3197$ and the equation to determine the critical θ_0 is

$$f\sqrt{\sin 2\theta} = 6\cdot3197-6\cdot3474\cos(\theta-7^\circ\ 19')-3\mathrm{F}(\theta)-\tfrac{1}{3}\mathrm{F}(\pi/2-\theta)$$

where $$\mathrm{F}(\theta) \equiv 0\cdot61108\cos\theta\log_{10}\cos\theta,$$

and f is a function of θ given by

$$\tan^2\tfrac{1}{2}\phi = 3^{2/3}\tan\theta.$$

There will be two roots, one corresponding to the configuration where the weaker ring is inside, the other to that in which it is outside. They are respectively very close to $\theta_0 = 2^\circ\,30'$ and $\theta_0 = 32^\circ$, corresponding to

$$b/a = 0{\cdot}037 \text{ (weaker inside),}$$

$$b/a = 0{\cdot}769 \text{ (weaker outside).}$$

Thus if the weaker is placed by chance inside the stronger it is practically certain to remain in union. If, however, the stronger is put by chance inside the weaker the chances are three to one against the partnership being permanent.

The curve C in fig. 5, represents the relative path of the weaker as seen from the stronger, when $\theta_0 = 15^\circ$ for weaker inside. It corresponds to ratio of radii of weak to strong of 0·557. For this case the value of C in the equation to the path is 6·4974.

11. *Opposite Circulations.*—With $\mu_2 = -\mu_2$ equation (5) becomes

$$\mu_1 x_1^2 - \mu_2 x_2^2 = \text{A}, \qquad (12)$$

with

$$\mu_1 x_1^2 + \mu_2 x_2^2 = \text{X}.$$

The former transformations may be replaced by*

$$x_1 = \sqrt{\frac{\text{A}}{\mu_1}} \cosh\theta, \qquad x_2 = \sqrt{\frac{\text{A}}{\mu_2}} \sinh\theta,$$

$$\mu_1 = \mu \cosh^2\alpha, \qquad \mu_2 = \mu \sinh^2\alpha,$$

$$(\mu_1 - \mu_2)\, d^2 = \mu d^2 = \text{A},$$

so that

$$x_1 = d\cosh\theta/\cosh\alpha, \qquad x_2 = d\sinh\theta/\sinh\alpha,$$

$$x = 2d\sinh(\alpha-\theta)/\sinh 2\alpha, \qquad \text{X} = \mu d^2\cosh 2\theta,$$

also when co-planar, the value of ϕ in f is given by

$$\tan^2\tfrac{1}{2}\phi = \coth\alpha\tanh\theta.$$

In fact, the formulæ obtained for μ_1, μ_2 positive are transformed by writing θi, αi.

In the case of like circulations, with $\mu_1 x_1^2 + \mu_2 x_2^2$ constant, it is clear that the apertures can never vary largely from those in the standard co-planar position. Here, however, any values up to infinity may enter so far as condition (12) is concerned. If the circulations are equal it is clear that $x = x_1 - x_2$ can never change sign, that is, one ring can never cross the path of the other. When the circulations are not equal, this crossing is possible but only when the two apertures have opened out to extremely large amounts, unless the two μ_1, μ_2 differ considerably. With equal μ, the θ, α, μ, etc., become indeterminate, and it is better to treat this case independently.

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* As existing tables are more complete for circular than for hyperbolic functions, it might be advisable to use the unsymmetrical transformation $\mu_1 = \mu\sec^2\alpha$, $\mu_2 = \mu\tan^2\alpha$, $x_1 = d\sec\theta/\sec\alpha$, $x_2 = d\tan\theta/\tan\alpha$, $\text{X} = \mu d^2(1+\sin^2\theta)/\cos^2\theta$, $x = 2d\cos\alpha\sec\theta(1-\sin\theta\operatorname{cosec}\alpha)$, $\tan^2\frac{1}{2}\phi = \sin\theta/\sin\alpha$.

12. *Conditions for Union with Opposite Circulations.*—Consider the critical co-planar state which is the boundary between configurations of permanent and temporary union. If the inner ring is slightly smaller it will pass through and on to infinity. If it is larger, its own velocity of progression is smaller than that due to the other ring and it is swept back, and its aperture still further increased by the other. In the critical condition, therefore, the two rings must just be at relative rest or $dy/dt = 0$ when $y = 0$. Hence

$$\phi\,(\mathrm{X}) - \tfrac{1}{2}\,\frac{\mathrm{X}}{x_1{}^2 x_2{}^2}\,\chi + \left\{-\tfrac{1}{2}\,\frac{\mathrm{X}}{x_1{}^2 x_2{}^2} + \frac{x_1+x_2}{4x_1{}^2 x_2{}^2}\,(\mu_1 x_1 + \mu_2 x_2)\,k^2\right\} k\,\frac{d\chi}{dk} = 0,$$

whilst the simultaneous condition $y = 0$ gives $k = \sin\phi$ from

$$\tan^2 \tfrac{1}{2}\phi = x_2/x_1.$$

In any given pair of rings μ_1/μ_2 and other constants are known, and this is an equation to determine x_1/x_2 in the critical configuration. Our present purpose, however, is to obtain information as to the general nature of the motions. It will be simpler, therefore, to take an instance of x_1, x_2 given and determine the values of $\mu_1/\mu_2 (\equiv z$ say), which will make this configuration a critical one. We will take

$$x_1 = nd, \qquad x_2 = d, \qquad \tan\tfrac{1}{2}\phi = n^{-1/2}, \qquad (n > 1).$$

The equation of condition may now be written

$$\mu_1 \left\{ nf + n\left(1 - \frac{n+1}{2n}k^2\right) k\,\frac{df}{dk} - \frac{3}{4\sqrt{n}}\log_e \lambda_1 x_1 \right\}$$
$$= \mu_2 \left\{ \frac{3\sqrt{n}}{4}\log\lambda_2 x_2 - f - \left(1 - \frac{n+1}{2n}k^2\right) k\,\frac{df}{dk} \right\}.$$

Here $\qquad \lambda_1 = \left(\dfrac{96\pi}{\sqrt{e}}\right)^{1/3}\dfrac{1}{c_1}, \qquad \lambda_2 = \left(\dfrac{96\pi}{\sqrt{e}}\right)^{1/3}\dfrac{1}{c_2}.$

Further if the vorticities of the two rings are the same

$$\mu_1/\mu_2 = (c_1/c_2)^3 = z,$$

$$\therefore \qquad \lambda_1 x_1 = \lambda_2 z^{-1/3} n\,.\,x_2 = \left(\frac{96\pi}{\sqrt{e}}\right)^{1/3} \frac{n}{z^{1/3}}\cdot\frac{d}{c_2}.$$

Write $\qquad \left(\dfrac{96\pi}{\sqrt{e}}\right)^{1/3} = 5{\cdot}676 = \lambda, \qquad f + \left(1 - \dfrac{n+1}{2n}k^2\right) k\,\dfrac{df}{dk} = \gamma.$

Then

$$z = n\,\frac{\log_{10}\lambda\,d/c - 0{\cdot}57905\,\gamma/\sqrt{n}}{0{\cdot}57905\,n\sqrt{n}\gamma - \log n + \frac{1}{3}\log z - \log_{10}\lambda\,d/c}.$$

In this

$$\gamma = f + \left(1 - \frac{n+1}{2n}k^2\right) k\,\frac{df}{dk} \text{ and}$$

$$k\,\frac{df}{dk} = -\frac{1}{k}\,\mathrm{F} + \frac{1+k'^2}{kk'^2}\,\mathrm{E}.$$

In general it is clear that for a positive value of z it is necessary that

$$\log\lambda_2 x_2 > \frac{4}{3\sqrt{n}}\gamma, \qquad > \frac{1}{n^2}\log\lambda_1 x_1,$$

or

$$(\lambda_2 x_2)^{n^2} > \lambda_1 x_1 \qquad > nz^{-1/3}\lambda_2 x_2,$$

$$\left(\lambda\frac{d}{c}\right)^{n^2-1} > nz^{-1/3}$$

For illustrative cases take $x_2/x_1 = 0\cdot81$ and $0\cdot5$.

$$x_2/x_1 = 0\cdot81, \qquad \tan\tfrac{1}{2}\phi = 0\cdot9, \qquad \phi = 84^\circ.$$

This gives $\gamma = 10\cdot715$ and

$$z = \frac{100}{81}\,\frac{\log\lambda\, d/c - 5\cdot584}{8\cdot475 + \frac{1}{3}\log z - \log\lambda\, d/c}.$$

Hence $$\lambda\, d/c > 10^{5\cdot6} \quad < 10^{8\cdot4}.$$

If $$\frac{\lambda d}{c} = 10^6, \qquad z = 1\cdot234\,\frac{0\cdot416}{2\cdot475 + \frac{1}{3}\log z} = 0\cdot22.$$

If $$\frac{\lambda d}{c} = 10^8, \qquad z = 1\cdot234\,\frac{2\cdot416}{0\cdot475 + \frac{1}{3}\log z} = 4\cdot3.$$

For the first the stronger is inside, for the second outside. For all possible cases the filament will be very thin, of order $10^{-7}\times$ the aperture.

$$x_2/x_1 = 0\cdot5, \qquad \phi = 70^\circ\, 32', \qquad \gamma = 4\cdot155,$$

$$z = 2\,\frac{\log\lambda\, d/c - 1\cdot701}{6\cdot504 + \frac{1}{3}\log z - \log\lambda\, d/c}$$

$$= 2\,\frac{\log d/c - 0\cdot947}{5\cdot750 + \frac{1}{3}\log z - \log d/c}.$$

For comparison with the cases considered with like circulations take $\log d/c = 3$,

$$z = 2\,\frac{2\cdot053}{2\cdot750 + \frac{1}{3}\log z} = 1\cdot46.$$

13. *Equal and Opposite Rings.*—In this case as we have seen one ring can never cross the path of the other, that is the ring with the smaller aperture at any instant will always have the smaller aperture. We consider first the configuration of equal apertures, when the apertures are then always equal. It is the case indeed in which a single ring moves towards a plane boundary which is parallel to its own plane. Refer the motion to the mid-plane (or the fixed boundary). Then the co-ordinates of the ring are $x = x_1 = x_2$ and $y = y_1 = -y_2$, and

$$k^2 = \sin^2\phi = \frac{x^2}{x^2+y^2}, \qquad \tan\phi = x/y.$$

Hence if O be the point where the central axis cuts the plane, and P the centre of the cross-section of the filament, OP $= r$ and OP makes an angle ϕ with the direction of translation. In fact r, ϕ are the polar co-ordinates of the position of the ring.

The equation to the path is given by

$$x\frac{df}{d\mathrm{X}} = \frac{3}{16\mu x}\log\lambda x - \frac{1}{4\pi x}f,$$

or since $\mathrm{X} = 2\mu x^2$

$$d/dx\,(xf) = \tfrac{3}{4}\log\lambda x,$$

whence

$$xf = l + \tfrac{3}{4}\,x\log\lambda x/e,$$

where l is a constant of integration. Also $x = r\sin\phi = rk$,

$$\therefore \qquad \frac{l}{r} = -\tfrac{3}{4}\,k\log\frac{\lambda rk}{e} + (1-\tfrac{1}{2}k^2)\,\mathrm{F}-\mathrm{E}.$$

When ϕ approaches $\frac{1}{2}\pi$ or k' approaches zero

$$\frac{l}{r} = -\tfrac{3}{4}\log\frac{\lambda r}{e} - 1 + \tfrac{1}{2}\log\frac{4}{k'}$$

$$= \tfrac{1}{4}\log\frac{16}{\lambda^3 e}\cdot\frac{1}{k'^2 r^3}.$$

Hence r becomes infinite at the rate

$$r^3 = \frac{16}{\lambda^3 e k'^2},$$

or

$$r = \frac{2}{\lambda}\left(\frac{2}{e}\right)^{1/3}\sec^{2/3}\phi = \mathrm{A}\sec^{2/3}\phi \text{ as } \phi \text{ tends to } \tfrac{1}{2}\pi,$$

also $y = r\cos\phi = \mathrm{A}\cos^{1/3}\phi$.

If c is the radius of the cross-section at any time

$$rc^2 = \text{const.} = r_0 c_0{}^2,$$

$$\frac{c}{y} = \frac{c_0 r_0{}^{1/2}}{\mathrm{A}^{3/2}} = \text{constant}.$$

If then c/y is sufficiently small at any time, *i.e.* the filament sufficiently thin, to justify our use of the formula for χ, the formula will hold even when the two rings become infinitely close when their apertures are infinitely large, and the equation to the path will remain throughout that given above, if no other conditions occur. But if the filaments decrease indefinitely a state will intervene, depending on the pressure of the fluid, at which a hollow is formed* and the limiting case is one of constant cross-section. Unless, therefore, the pressure is infinitely large some secondary changes will intervene after a certain point, the cross-sections will not be circular, and the configuration altered in some way not hitherto determined.

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* 'Proc. Camb. Phil. Soc.,' vol. 3, p. 284 (1879).

14. *Path in the fluid.*—Write $y_1+y_2 = \mathrm{Y}$. Then

$$\frac{d\mathrm{Y}}{dt} = \psi(\mathrm{X})+\frac{1}{2}\frac{\mathrm{A}}{x_1{}^2x_2{}^2}\chi+\left\{\frac{1}{2}\frac{\mathrm{A}}{x_1{}^2x_2{}^2}-\frac{(x_1+x_2)}{4x_1{}^2x_2{}^2}(\mu_1x_1+\mu_2x_2)k^2\right\}k\frac{dX}{dk}$$

$$= \psi(\mathrm{X})+\frac{1}{2}\frac{\mathrm{A}}{x_1{}^2x_2{}^2}\chi+\left\{\frac{1}{2}\frac{\mathrm{A}}{x_1{}^2x_2{}^2}-\frac{\mathrm{A}+(\mu_1+\mu_2)x_1x_2}{4x_1{}^2x_2{}^2}k^2\right\}k\frac{d\chi}{dk},$$

where
$$\psi(\mathrm{X}) = \frac{3\mu_1}{8x_1}\log\lambda_1x_1+\frac{3\mu_2}{8x_2}\log\lambda_2x_2.$$

Also (6)
$$\frac{d\mathrm{X}}{dt} = \mu_1\mu_2\frac{k^3y}{x_1x_2}\frac{d\chi}{dk},$$

$$\therefore \quad \frac{d\mathrm{Y}}{d\mathrm{X}} = \frac{\psi(\mathrm{X})}{\mu_1\mu_2k^3}\frac{\sqrt{(x_1x_2)}}{y}\cdot\frac{1}{df/dk}+\frac{\mathrm{A}}{2\mu_1\mu_2x_1x_2y}\cdot\frac{f}{df/dk}$$
$$+\frac{\mathrm{A}}{2\mu_1\mu_2x_1x_2y}\left\{\frac{1}{k^2}-\frac{1+(\mu_1+\mu_2)x_1x_2}{2\mathrm{A}}\right\}.$$

It is, of course, not integrable directly, but in any special case in tracing the relative path, contemporaneous values of k, y, and X (or θ) are known. The curve representing the right hand in terms of X as abscissa can consequently be traced, and Y obtained in terms of X by mechanical quadratures.

15. *The Time.*—The relation to the time again cannot be found in general terms, but it may be found for special cases by (6)

or
$$\frac{dt}{d\mathrm{X}} = \frac{\sqrt{(x_1x_2)}}{\mu_1\mu_2k^3y}\frac{1}{df/dk}$$

in the same way as Y in the previous article.

Table I.

θ.	$0\cdot61108\cos\theta\log_{10}\cos\theta$.	$0\cdot61108\sin\theta\log_{10}\sin\theta$.
°		
5	−0·00100	−0·05631
10	−0·00400	−0·08068
15	−0·00883	−0·09284
20	−0·01551	−0·09738
25	−0·02366	−0·09660
30	−0·03306	−0·09198
35	−0·04336	−0·08461
40	−0·05418	−0·07539
45	−0·06504	−0·06504

Table II.

$f = (\mathrm{F}-\mathrm{E})/k - \frac{1}{2}k\mathrm{F}, \qquad k = \sin\phi.$

ϕ.	f.	ϕ.	f.
0°	0·	55	0·11398
10	0·00049	60	0·1579
15	0·00180	65	0·2172
20	0·00426	70	0·2982
30	0·01515	75	0·4146
35	0·02461	80	0·5932
40	0·03693	85	0·9213
45	0·05647	89	1·7199
50	0·08106	90	∞

A Spectroscopic Investigation of the Ionisation of Argon by Electron Collisions.

By Frank Horton, Sc.D., Professor of Physics in the University of London, and Ann Catherine Davies, D.Sc., Royal Holloway College.

(Communicated by Sir J. J. Thomson, O.M., F.R.S. Received June 17, 1922.)

Introduction.

The authors have previously described an investigation of the effects of electron collisions with argon atoms,* in which the production of radiation and of ionisation was looked for, using a delicate electrometer as the detecting instrument. As the result of these experiments, it was found that the value of the minimum electron energy for the production of resonance radiation from argon was 11·5 volts, and the minimum electron energy necessary for the detachment of a single electron from a normal argon atom was 15·1 volts. Since the publication of these results some other investigations of the critical voltages for electrons in argon have been published. Several of these were made using argon as a gas filling for a thermionic valve, but the precise methods of arranging the electric fields, and of carrying out the experiment differed somewhat in different cases. The values of the ionisation voltage deduced by the various observers are not in good agreement. For instance, from experiments with three-electrode valves, Stead and Gossling† concluded that this value is 12·5 volts, Hodgson and Palmer‡ that it is 16·6 volts, and

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* F. Horton and A. C. Davies, 'Roy. Soc. Proc.,' A, vol. 97, p. 1 (1920).

† G. Stead and B. S. Gossling, 'Phil. Mag.,' vol. 40, p. 413 (1920).

‡ B. Hodgson and L. S. Palmer, 'Rad. Rev.,' vol. 1, No. 11 (1920).